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Radiation and diffraction of second-order surface waves by floating bodies

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The paper studies the radiation and diffraction by floating bodies of deep-water bichromatic and bidirectional surface waves subject to the second-order free-surface condition. A theory is developed for the evaluation of the second-order velocity potential and wave forces valid for bodies of arbitrary geometry, which does not involve the evaluation of integrals over the free surface or require an increased accuracy in the solution of the linear problem. Explicit sum- and difference-frequency 'Green functions' are derived for the radiation and diffraction problems, obtained from the solution of initial-value problems that ensure they satisfy the proper radiation condition at infinity. The second-order velocity potential is expressed as the sum of a particular and a homogeneous component. The former satisfies the non-homogeneous free-surface condition and is expressed explicitly in terms of the second-order Green functions. The latter is subject to the homogeneous free-surface condition and enforces the body boundary condition by the solution of a linear problem. An analysis is carried out of the singular behaviour of the second-order potential near the intersection of the body boundary with the free surface.

1. Introduction

The prediction of the wave loads and responses of offshore structures requires in principle the solution of complex nonlinear surface-wave radiation and diffraction problems. The volume and complexity of the associated computational effort encouraged their alternative treatment by a perturbation series expansion with respect to the wave slope. The leading term in such a series leads to the classical linear wave–body interaction problem, which has received extensive analytical and numerical treatment and has proven a valuable and reliable tool in practice. The linear theory is reviewed by Wehausen (1971) and Newman (1983), and its numerical aspects by Yeung (1982).

Quantities of practical interest which cannot be accounted for by linear theory include the slowly-varying hydrodynamic excitation responsible for large excursions of bodies constrained by weak restoring forces, and the rapidly varying hydrodynamic pressure force which contributes to the fatigue of offshore structures. These effects can be evaluated by the solution of the second-order problem in which the free-surface condition is non-homogeneous. The forcing terms involve quadratic products of the linear velocity potential and its spatial derivatives, and extend over the entire free surface. The solution of the resulting boundary-value problem is a substantially more complex task relative to its linear counterpart because of the infinite extent of the non-homogeneity of the free-surface condition and the approximate knowledge of the linear solution on the free surface. The second-order wave–body interaction theory is reviewed by Ogilvie (1983).

Several existing numerical solutions of the second-order radiation and diffraction problems are based on the application of Green's identity using the linear wave-source potential or the Rankine source $1/r$ as the Green function. Either choice leads to the evaluation of slowly convergent integrals over the free surface involving products of the forcing term in the second-order free-surface condition with the Green function. Their computation in three dimensions is time consuming and requires accurate values of the linear velocity potential and its gradients.

The present paper develops an alternative methodology for the solution of the second-order bichromatic radiation and diffraction problems which does not require the evaluation of infinite free-surface integrals. It is valid for bodies of arbitrary geometry and its numerical implementation is easy to carry out using existing boundary-integral methods. The method hinges on the derivation of two second-order Green functions which can be expressed in the form of explicit Fourier integrals, analogous to that in the definition of the linear wave-source potential (Wehausen & Laitone 1960, equation (13.15)). The diffraction Green function is the velocity potential which represents the second-order interaction of a regular plane progressive wave of frequency ω_0 with a submerged point source pulsating in a time-harmonic manner at a frequency ω_1. The radiation Green function is the velocity potential representing the second-order interaction of two submerged point sources placed at two different locations and pulsating at two different frequencies ω_1 and ω_2. Explicit solutions of these second-order problems are derived in the time domain, and their sum- and difference-frequency time-harmonic limits are obtained in the limit as time tends to infinity. This process ensures that they satisfy the proper radiation condition at infinity. The derivation of the Green functions is presented in §3.

The radiation and diffraction Green functions permit the construction of an explicit particular solution of the second-order problem. It is obtained in terms of the known linear radiation and diffraction velocity potentials on the body boundary and the second-order Green functions. It satisfies the non-homogeneous free-surface condition but not the normal velocity condition on the body boundary. The latter is enforced by the addition of a homogeneous component subject to the linear free-surface condition which can be determined using standard boundary-integral methods of linear theory. The sum of the particular and homogeneous components is the complete solution of the second-order velocity potential. The particular solution is regular on the body boundary and its interior and can be regarded as an incident potential flow which interacts with the body generating the homogeneous solution. If only the second-order forces are required, reciprocity relations directly analogous to the Haskind relations in linear theory can be derived. They involve integrals of the particular solution and an auxiliary linear potential over the body boundary. Here, the particular solution being harmonic in the entire fluid domain plays the role of the ambient regular waves in the linear problem. The formulation of the second-order problem is presented in §4 and its solution is given in §5.

For bodies which pierce the free surface, the second-order velocity potential develops a singular behaviour in the vicinity of the body waterline. This behaviour is analysed in §5 for a model two-dimensional problem for an intersection angle of 90°. It is shown that the second-order potential forced by the linear diffraction problem is analytic at the intersection. The second-order radiation potential, on the other hand, develops a singularity when forced by the linear sway and roll potentials, but remains analytic when forced only by the linear heave potential. In the former case, the value of the radiation potential at the intersection is finite but its vertical derivative is infinite. Moreover, it is shown that the forcing terms in the second-order free-surface condition develop a logarithmic singularity at the intersection which may require careful treatment in numerical solutions which integrate the free-surface inhomogeneity by quadrature.

The radiation condition in the present method of solution of the second-order problem is imbedded in its particular and homogeneous components. The radiation condition obeyed by the former is similar to the far-field behaviour of the second-order radiation and diffraction Green functions. The homogeneous solution is obtained from the solution of a linear problem, therefore it satisfies the Sommerfeld radiation condition at infinity. The radiation condition in the monochromatic diffraction problem has been studied by Molin (1979). A more complete radiation condition for the bichromatic second-order radiation and diffraction problems, including terms not accounted for by Molin, is derived by Wang (1987).

In §6 expressions are derived for the forces (moments are understood hereinafter) corresponding to the component of the second-order problem driven by the non-homogeneity of the free-surface condition. They are based on both the direct integration of the hydrodynamic pressure over the body boundary as well as the use of reciprocity relations similar to the Haskind exciting forces in linear theory. The latter express the forces in terms of the particular solution and an auxiliary linear potential. The forces corresponding to the component of the problem subject to the second-order body boundary condition and the linear free-surface condition can also be obtained either by direct pressure integration or by the application of the reciprocity relations. In the latter case, it is shown that the double gradients of the linear velocity potentials present in the second-order body boundary condition can be reduced to single gradients by virtue of a vector theorem used in the evaluation of the hydrodynamic coefficients in ship motion theory by Ogilvie & Tuck (1969). This reduction in the order of the spatial derivatives of the linear solution which must be evaluated on the body boundary is a desirable property in connection with numerical solutions based on boundary integral methods.

The principal computational task in the implementation of the present theory is the definition of the particular solution, or equivalently the evaluation of the second-order radiation and diffraction Green functions. The determination of the homogeneous component requires an effort comparable to that associated with the solution of a linear problem. By comparison to existing numerical solutions, knowledge is required only of the linear velocity potential over the body boundary with an accuracy which is not expected to exceed that of the linear quantities.

The present theory can be generalized to the case where the water depth is finite by deriving the finite-depth second-order Green functions. This as well as further extensions, are discussed in §7.

2. Formulation – the linear Green function

Figure 1 illustrates a Cartesian coordinate system (x, y, z) fixed relative to the undisturbed position of the free surface ($z = 0$ plane, with the z-axis pointing upwards. A potential flow is assumed, governed by the velocity potential $\Phi(\boldsymbol{x}, t)$ which satisfies the Laplace equation in the fluid domain and the nonlinear free-surface condition

$$\Phi_{tt} + g\Phi_z + 2\nabla\Phi\cdot\nabla\Phi_t + \tfrac{1}{2}\nabla\Phi\cdot\nabla(\nabla\Phi\cdot\nabla\Phi) = 0, \tag{2.1}$$

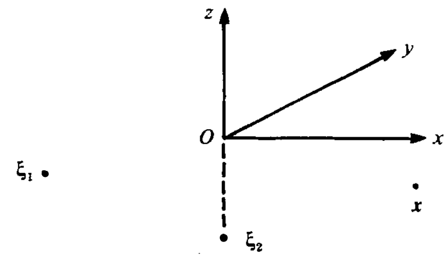

FIGURE 1

applied on

$$\zeta(x, y) = -\frac{1}{g}(\Phi_t + \tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi)_{z=\zeta}. \tag{2.2}$$

Postulating the existence of the perturbation series expansion for Φ and ζ,

$$\Phi = \Phi^{(1)} + \Phi^{(2)} + \dots, \tag{2.3}$$

$$\zeta = \zeta^{(1)} + \zeta^{(2)} + \dots \tag{2.4}$$

the leading term in the series satisfies the linear free-surface condition obtained by Taylor expanding (2.1) and (2.2) around the calm-water position $z = 0$

$$\Phi_{tt}^{(1)} + g\Phi_z^{(1)} = 0, \quad \text{on } z = 0, \tag{2.5}$$

$$\zeta^{(1)} = -\frac{1}{g}\Phi_t^{(1)}|_{z=0}. \tag{2.6}$$

The free-surface condition (2.5) must be supplemented by initial conditions appropriate to the problem being studied. We consider here a linear wave disturbance of the form

$$\Phi^{(1)} = \Phi_0 + \Phi_1 + \Phi_2. \tag{2.7}$$

The first component is a deep-water regular plane progressive wave persisting for all time and defined by

$$\Phi_0 = \mathrm{Re}\,(\varphi_0\,\mathrm{e}^{\mathrm{i}\omega_0 t}), \quad \varphi_0 = \frac{\mathrm{i}gA}{\omega_0}\,\mathrm{e}^{\nu_0 z - \mathrm{i}\nu_0 x + \mathrm{i}\omega_0 t}, \tag{2.8a, b}$$

where g is the acceleration due to gravity, A is the wave amplitude, ω_0 is its frequency and $\nu_0 = \omega_0^2/g$. The second and third components in (2.7) are the velocity potentials of two sources located at $(\xi_i, \eta_i, \zeta_i), i = 1, 2$ which start pulsating from rest with strengths $\cos\omega_i t, i = 1, 2$. The respective velocity potentials are subject to the free-surface condition (2.5) the initial conditions of zero wave elevation and zero pressure-impulse on the free surface, and are given by (Stoker 1957)

$$\Phi_i = \left(\frac{1}{r} - \frac{1}{r'}\right) + \frac{1}{\pi}\,\mathrm{Re}\int_0^\infty \mathrm{d}k (gk)^{\frac{1}{2}} \int_0^{2\pi} \mathrm{d}\theta\, \mathrm{e}^{k(z+\zeta_i) + \mathrm{i}k(x-\xi_i)\cos\theta + (y-\eta_i)\sin\theta} \times \int_0^t \mathrm{d}\tau \sin((gk)^{\frac{1}{2}}(t-\tau))\,\mathrm{e}^{\mathrm{i}\omega_i\tau}, \tag{2.9}$$

where

$$r = ((x-\xi_i)^2 + (y-\eta_i)^2 + (z-\zeta_i)^2)^{\frac{1}{2}}, \quad r' = ((x-\xi_i)^2 + (y-\eta_i)^2 + (z+\zeta_i)^2)^{\frac{1}{2}}.$$

The time-convolution integral in (2.9) can be evaluated explicitly in the form

$$\int_0^t \mathrm{d}\tau \sin((gk)^{\frac{1}{2}}(t-\tau))\,\mathrm{e}^{\mathrm{i}\omega\tau} = \mathrm{e}^{\mathrm{i}\omega t}\left(\frac{(gk)^{\frac{1}{2}}}{gk-\omega^2} - \frac{1}{2}\frac{\mathrm{e}^{\mathrm{i}((gk)^{\frac{1}{2}}-\omega)t}}{(gk)^{\frac{1}{2}}-\omega} - \frac{1}{2}\frac{\mathrm{e}^{-\mathrm{i}((gk)^{\frac{1}{2}}+\omega)t}}{(gk)^{\frac{1}{2}}+\omega}\right). \tag{2.10}$$

Upon substitution of (2.10) into (2.9), the k-integration over the pole at $k = \omega^2/g$ is interpreted in the principal-value sense. The use of (2.10) in (2.9) allows the separation of the transient from the steady-state time-harmonic components. Here, use will be made of the following Lemma from the theory of Fourier integrals:

LEMMA 1.

If the real function $f(k)$ has a positive zero at $k = a$ and the function $F(k)$ is regular in $[0, \infty)$, then as $t \to \infty$

$$PV \int_0^\infty F(k) \frac{\mathrm{e}^{\mathrm{i}tf(k)}}{f(k)} = \pi i \frac{F(a)}{|f'(a)|} + O(1/t). \tag{2.11}$$

If $f(k)$ has no zeros in the range of integration, then the integral is of $O(1/t)$, as $t \to \infty$.

Substituting (2.10) into (2.9), letting $t \to \infty$ and making use of Lemma 1, the wave source potential reduces to the form

$$\Phi_i = \Phi_{iS} + \Phi_{iT} \tag{2.12}$$

with $\Phi_{iT} = O(1/t)$ as $t \to \infty$, and

$$\Phi_{iS} = \mathrm{Re}\,(G_i\, \mathrm{e}^{\mathrm{i}\omega_i t}), \tag{2.13a}$$

$$G_i(\boldsymbol{x}; \xi_i) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{\mathrm{d}u\,\mathrm{d}v}{(u^2+v^2)^{\frac{1}{2}}} \left(\frac{(u^2+v^2)^{\frac{1}{2}} + \nu_i}{(u^2+v^2)^{\frac{1}{2}} - \nu_i^\epsilon} \mathrm{e}^{(z+\zeta_i)(u^2+v^2)^{\frac{1}{2}}} + \mathrm{e}^{-|z-\zeta_i|(u^2+v^2)^{\frac{1}{2}}} \right) \times \mathrm{e}^{\mathrm{i}u(x-\xi_i)+\mathrm{i}v(y-\eta_i)}, \tag{2.13b}$$

where $\nu_i^\epsilon = (\omega_i - \mathrm{i}\epsilon)^2/g$. The shift of the wavenumber $\nu_i = \omega_i^2/g$ in the negative imaginary plane by a small quantity ϵ enforces the deviation of the k-contour of integration above the pole at $k = \nu_i$ in the limit as $\epsilon \to 0$. This is suggested by the contribution from the second term in the right-hand side of (2.10), following the application of Lemma 1. This choice of the k-integration ensures the validity of the Sommerfeld radiation condition in the linear problem.

The second-order interaction of the velocity potentials Φ_1 and Φ_2 defines the radiation Green function, while the interaction of Φ_0 and Φ_1 or Φ_2 defines the diffraction Green function. They are derived in the next section.

3. The second-order Green functions

This section presents the derivation of the time-harmonic radiation and diffraction second-order 'Green functions'. They can be more accurately described by the name 'elementary second-order wave-source potentials'. The former name has been adopted for the sake of brevity and convenience. Conventional Green functions are singular at the location of the source and usually satisfy all boundary conditions of the boundary-value problem being solved, except for the body boundary condition. Neither of these two conditions are met by the present elementary solutions. They are regular at the location of the source, and only an appropriate linear combination of them satisfies the second-order free-surface condition. Otherwise, they are subject to the Laplace equation in the fluid domain and can be expressed in the form of explicit Fourier integrals. They are obtained from the solution of associated initial-value problems by allowing time to approach infinity. As in the linear problem, this approach allows the derivation of the proper radiation condition for the second-order velocity potentials. The analysis which allows the transition from the transient to the time-harmonic disturbance is outlined first, and is followed by the definition of the sum- and difference-frequency Green functions.

Keeping quadratic terms in the Taylor-series expansion of the free-surface conditions (2.1) and (2.2) around $z=0$, we obtain for the second-order velocity potential

$$\Phi_{tt}^{(2)}+g\Phi_z^{(2)} = Q(x,y,t) \quad \text{on } z=0, \tag{3.1}$$

$$Q(x,y,t) = -\frac{\partial}{\partial t}(\nabla\Phi^{(1)}\cdot\nabla\Phi^{(1)})+\frac{1}{g}\Phi_t^{(1)}\frac{\partial}{\partial z}(\Phi_{tt}^{(1)}+g\Phi_z^{(1)}), \tag{3.2}$$

where the linear velocity potential $\Phi^{(1)}$ is defined by (2.7), (2.8) and (2.9). The substitution of the linear velocity potential (2.7) in (3.2) generates three contributions to the second-order potential $\Phi^{(2)}$. Quadratic products of Φ_0 define a boundary-value problem for the second-order correction to the linear incident waves. For deep-water monochromatic waves this correction is known to vanish. This can be verified by substituting (2.8) in (3.2). For bichromatic or bidirectional regular waves second-order corrections exist. Their form in deep water is derived in §4. A discussion of their properties in water of finite and infinite depth is presented by Ogilvie (1983). Quadratic products of the transient wave source disturbance (2.9) define a second-order problem for the radiation Green function. Cross products of the incident-wave and either of the wave source disturbances define the second-order problem for the diffraction Green function.

Assume that the second-order velocity potential $\Phi^{(2)}$ is subject to the initial conditions

$$\Phi^{(2)}(x,y,0,0^+) = A(x,y), \quad \Phi_t^{(2)}(x,y,0,0^+) = B(x,y), \tag{3.3}$$

applied on the $(z=0)$-plane. In the transient problem studied here and for finite t, the functions $Q(x,y,t)$, $A(x,y)$ and $B(x,y)$ decay sufficiently rapidly as $R=(x^2+y^2)\rightarrow\infty$ for their Fourier transform with respect to the (x,y) coordinates to exist. Let

$$\tilde{\Phi}(u,v) = \iint_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{d}y\; \mathrm{e}^{\mathrm{i}ux+\mathrm{i}vy}\,\Phi(x,y). \tag{3.4}$$

Fourier transforming (3.1) and dropping the superscripts in the second-order potential, we obtain

$$\tilde{\Phi}_{tt}(u,v,z=0,t)+g\tilde{\Phi}_z(u,v,z=0,t) = \tilde{Q}(u,v,t). \tag{3.5}$$

Since Φ satisfies the three-dimensional Laplace equation, $\tilde{\Phi}$ is subject to

$$\tilde{\Phi}_{zz}-(u^2+v^2)\,\tilde{\Phi} = 0. \tag{3.6}$$

Solutions of (3.6) which vanish as $z\rightarrow-\infty$ are of the form

$$\tilde{\Phi}(u,v;z,t) = C(u,v;t)\;\mathrm{e}^{z(u^2+v^2)^{\frac{1}{2}}}. \tag{3.7}$$

Substituting (3.7) into (3.5), we obtain the differential equation for $C(u,v;t)$

$$C(u,v;t)_{tt}+g(u^2+v^2)^{\frac{1}{2}}C(u,v;t) = \tilde{Q}(u,v;t), \tag{3.8}$$

subject to the initial conditions

$$\tilde{C}(u,v;0) = \tilde{A}(u,v), \quad \tilde{C}_t(u,v;0) = \tilde{B}(u,v). \tag{3.9}$$

The solution to the initial-value problem (3.8)–(3.9) is

$$C(u,v;t) = \tilde{A}\cos((gk)^{\frac{1}{2}}t)+\frac{1}{(gk)^{\frac{1}{2}}}\left(\tilde{B}\sin((gk)^{\frac{1}{2}}t)+\int_0^t \mathrm{d}\tau\,\sin((gk)^{\frac{1}{2}}(t-\tau))\,\tilde{Q}(u,v;\tau)\right), \tag{3.10}$$

where $k = (u^2+v^2)^{\frac{1}{2}}$. Substituting (3.10) into (3.7) and inverting the double Fourier transform, we obtain

$$\Phi(x,y,z;t) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \mathrm{d}u\,\mathrm{d}v \frac{\mathrm{e}^{kz-\mathrm{i}ux-\mathrm{i}vy}}{(gk)^{\frac{1}{2}}}\bigg(\tilde{A}(gk)^{\frac{1}{2}}\cos((gk)^{\frac{1}{2}}t)$$
$$+\tilde{B}\sin((gk)^{\frac{1}{2}}t)+\int_0^t \mathrm{d}\tau\,\sin((gk)^{\frac{1}{2}}(t-\tau))\,\tilde{Q}(u,v;\tau)\bigg). \quad (3.11)$$

Equation (3.11) is the solution to the second-order transient problem (3.1) and (3.3). The limiting behaviour of (3.11) as $t\rightarrow\infty$ is studied next.

An inspection of (3.10)–(3.11) suggests that the first two terms in (3.10) arising from the initial conditions contribute decaying transients to the second-order potential. This follows easily by an integration by parts of (3.11). Denote by $Q_{\mathrm{T}}(x,y,t)$ the parts of Q containing the transient component of the linear wave source potential (2.9). Let

$$I(u,v;t) = \int_0^t \mathrm{d}\tau\,\sin((gk)^{\frac{1}{2}}(t-\tau))\,\tilde{Q}_{\mathrm{T}}(u,v;\tau) = \mathrm{Im}\,(\mathrm{e}^{\mathrm{i}t(gk)^{\frac{1}{2}}}J(u,v;t)), \quad (3.12)$$

where
$$J(u,v;t) = \int_0^t \mathrm{d}\tau\,\mathrm{e}^{-\mathrm{i}\tau(gk)^{\frac{1}{2}}}\tilde{Q}_{\mathrm{T}}(u,v;\tau). \quad (3.13)$$

Since $\tilde{Q}_{\mathrm{T}}$ is finite at $t=0$ and of $O(1/t)$ as $t\rightarrow\infty$, $J(u,v;t)$ has no poles for real positive values of u,v and for any positive value of t. Combining this behaviour with (3.12) and substituting in (3.11), we conclude using Lemma 1 that the contribution of the transient component of the linear solution to the second-order potential is also transient, decaying like $O(1/t)$ as $t\rightarrow\infty$.

The transient component of the forcing function $Q(x,y,t)$ is hereinafter omitted. Invoking the quadratic dependence of Q on the linear solution, the time dependence of the incident-wave and source potentials (2.8) and (2.13), and making use of the identity

$$\mathrm{Re}\,(A_1)\,\mathrm{Re}\,(A_2) = \tfrac{1}{2}\mathrm{Re}\,(A_1A_2+A_1A_2^*), \quad (3.14)$$

where the * denotes complex conjugation, we may decompose the steady-state component of the forcing function Q into sum- and difference-frequency components

$$Q(x,y,t) = \mathrm{Re}\,(Q^+(x,y)\,\mathrm{e}^{\mathrm{i}(\omega_m+\omega_n)t}+Q^-(x,y)\,\mathrm{e}^{\mathrm{i}(\omega_m-\omega_n)t}), \quad (3.15)$$

where $m,n = 0,1,2$. The substitution of (3.15) into (3.11) leads to the evaluation of integrals similar to (2.10). Here, the frequency $\Omega^- = \omega_m-\omega_n$ is negative when $\omega_m<\omega_n$. Substituting (3.15) into (3.11) and allowing t to approach infinity, it follows that the contribution from the three terms in (2.10) is the same as in the linear problem when $\Omega^\pm>0$. When $\Omega^-<0$, the role of the second and third terms in the right-hand side of (2.10) are reversed. The second term contributes a transient and the third term generates the shift of the pole at $k=\Omega^2/g$ in the complex plane. This shift is, however, of opposite sign relative to the shift when $\Omega^\pm>0$. This follows from the negative sign of the argument of the exponential in the last term of (2.10) which leads to the application of the complex conjugate of Lemma 1. Therefore, the complex pole generated from the evaluation of the convolution integral in (3.11) in the limit $t\rightarrow\infty$, is defined by

$$N_\delta^\pm = [\omega_m\pm\omega_n-\mathrm{i}\,\mathrm{sgn}\,(\omega_m\pm\omega_n)\,\delta]^2/g, \quad (3.16)$$

where δ is a small positive parameter. The decomposition (3.15) suggests a similar decomposition for the second-order potential

$$\Phi(x, y, z, t) = \mathrm{Re}\,(\varphi^{+}(x, y, z)\, \mathrm{e}^{\mathrm{i}(\omega_m+\omega_n)t} + \varphi^{-}(x, y, z)\, \mathrm{e}^{\mathrm{i}(\omega_m-\omega_n)t}), \tag{3.17}$$

where the complex velocity potentials $\varphi^{\pm}$ follow from (3.11) in the form

$$\varphi^{\pm}(x, y, z) = \frac{1}{8\pi^2 g} \iint_{-\infty}^{\infty} \mathrm{d}u\, \mathrm{d}v\, \frac{\mathrm{e}^{z(u^2+v^2)^{\frac{1}{2}} - \mathrm{i}ux - \mathrm{i}vy}}{(u^2+v^2)^{\frac{1}{2}} - N_{\delta}^{\pm}}\, \tilde{Q}^{\pm}(u, v). \tag{3.18}$$

Equation (3.18) provides an explicit expression for the steady-state time-harmonic limit of the second-order solution. The radiation and diffraction Green functions will be obtained by supplying (3.18) with the functions $\tilde{Q}^{\pm}$ corresponding to each.

The two-dimensional version of the solution (3.18) is obtained by an integration over the entire y-axis and use of the identity

$$\int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{\mathrm{i}vy} = 2\pi\, \delta(v), \tag{3.19}$$

leading to

$$\varphi_{2\mathrm{D}}^{\pm}(x, y) = \frac{1}{4\pi g} \int_{-\infty}^{\infty} \mathrm{d}u\, \frac{\mathrm{e}^{|u|z - \mathrm{i}ux}}{|u| - N_{\delta}^{\pm}}\, \tilde{q}^{\pm}(u), \tag{3.20}$$

where $\tilde{q}^{\pm}(u)$ is the two-dimensional analogue of $\tilde{Q}^{\pm}(u, v)$.

The derivation of the three- and two-dimensional Fourier transforms is presented in the remainder of this section.

3.1. *The radiation Green functions*

3.1.1. *Sum-frequency problem*

Substituting the definitions of the time-harmonic wave-source potentials G_1 and G_2 (equation (2.13)) in (3.2), we can identify the characteristic forcing terms

$$\begin{aligned} Q_A^{+}(x, y) &= -\mathrm{i}(\omega_1+\omega_2)\, \nabla G_1 \cdot \nabla G_2 \\ &= \frac{-\mathrm{i}(\omega_1+\omega_2)}{\pi^2} \iint_{-\infty}^{\infty} \mathrm{d}u_1\, \mathrm{d}v_1 \iint_{-\infty}^{\infty} \mathrm{d}u_2\, \mathrm{d}v_2\, \frac{\mathrm{e}^{(k_1\zeta_1+k_2\zeta_2)}\, (\nu_1\nu_2 - u_1u_2 - v_1v_2)}{(k_1-\nu_1^{\epsilon})\,(k_2-\nu_2^{\epsilon})} \\ &\quad \times \mathrm{e}^{-\mathrm{i}(u_1\xi_1+u_2\xi_2) - \mathrm{i}(v_1\eta_1+v_2\eta_2)}\, \mathrm{e}^{\mathrm{i}x(u_1+u_2)+\mathrm{i}y(v_1+v_2)}, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} Q_B^{+}(x, y) &= \mathrm{i}\omega_1 G_1 \frac{\partial}{\partial z}(G_{2z} - \nu_2 G_2) \\ &= \frac{\mathrm{i}\omega_1}{\pi^2} \iint_{-\infty}^{\infty} \mathrm{d}u_1\, \mathrm{d}v_1 \iint_{-\infty}^{\infty} \mathrm{d}u_2\, \mathrm{d}v_2\, \frac{\mathrm{e}^{(k_1\zeta_1+k_2\zeta_2)}\, (k_2+\nu_2)}{k_1-\nu_1^{\epsilon}} \\ &\quad \times \mathrm{e}^{-\mathrm{i}(u_1\xi_1+u_2\xi_2) - \mathrm{i}(v_1\eta_1+v_2\eta_2)}\, \mathrm{e}^{\mathrm{i}x(u_1+u_2)+\mathrm{i}y(v_1+v_2)}, \end{aligned} \tag{3.22}$$

where $k_i = (u_i^2+v_i^2)^{\frac{1}{2}}$.

The Fourier transforms of (3.21) and (3.22) are evaluated by using the identity

$$\iint_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{d}y\, \mathrm{e}^{\mathrm{i}ux+\mathrm{i}vy} = 4\pi^2 \delta(u)\, \delta(v), \tag{3.23}$$

where $\delta(x)$ is the Dirac delta function. To simplify the algebra, and without loss of generality, we will assume that the origin of the coordinate system is located above the position of the source with index $i = 2$, so that $\xi_2 = \eta_2 = 0$ with $\zeta_2 < 0$ (see figure 1). It follows that

$$\tilde{Q}_A^+(u,v) = -4\mathrm{i}(\omega_1+\omega_2)\iint_{-\infty}^{\infty} \mathrm{d}u_1\,\mathrm{d}v_1 \iint_{-\infty}^{\infty} \mathrm{d}u_2\,\mathrm{d}v_2 \frac{\mathrm{e}^{(k_1\zeta_1+k_2\zeta_2)}\,(\nu_1\nu_2-u_1u_2-v_1v_2)}{(k_1-\nu_1^\epsilon)(k_2-\nu_2^\epsilon)}$$
$$\times\,\mathrm{e}^{-\mathrm{i}(u_1\xi_1+v_1\eta_1)}\,\delta(u+u_1+u_2)\,\delta(v+v_1+v_2)$$
$$= -4\mathrm{i}(\omega_1+\omega_2)\iint_{-\infty}^{\infty} \mathrm{d}u_1\,\mathrm{d}v_1 \frac{\mathrm{e}^{\zeta_1k_1+\zeta_2((u+u_1)^2+(v+v_1)^2)^{\frac{1}{2}}}\,(\nu_1\nu_2+k_1^2+uu_1+vv_1)}{(k_1-\nu_1^\epsilon)\,(((u+u_1)^2+(v+v_1)^2)^{\frac{1}{2}}-\nu_2^\epsilon)}$$
$$\times\,\mathrm{e}^{-\mathrm{i}(u_1\xi_1+\mathrm{i}v_1\eta_1)}. \tag{3.24}$$

An alternative form for (3.24) which may be more convenient for computations can be obtained by a transformation from the Cartesian to the polar wavenumber coordinate system, making use of the definitions

$$u = k\cos\phi, \quad v = k\sin\phi, \tag{3.25}$$

$$u_1 = l\cos\psi, \quad v_1 = l\sin\psi, \tag{3.26}$$

$$\rho(l,k,\chi) = (l^2+2lk\cos\chi+k^2)^{\frac{1}{2}}, \tag{3.27}$$

$$\xi_1 = R_{12}\cos\alpha_{12}, \tag{3.28}$$

$$\eta_1 = R_{12}\sin\alpha_{12}, \tag{3.29}$$

$$\tilde{Q}_A^+(k,\phi) = -4\mathrm{i}(\omega_1+\omega_2)\int_0^{2\pi}\mathrm{d}\psi\int_0^{\infty}\mathrm{d}l\,l\frac{[\nu_1\nu_2+l^2+kl\cos(\phi-\psi)]}{(l-\nu_1^\epsilon)\,[\rho(l,k,\phi-\psi)-\nu_2^\epsilon]}$$
$$\times\,\mathrm{e}^{\zeta_1l+\zeta_2\rho(l,k,\phi-\psi)-\mathrm{i}lR_{12}\cos(\alpha_{12}-\psi)}. \tag{3.30}$$

A similar analysis for $\tilde{Q}_B^+(k,\phi)$ leads to

$$\tilde{Q}_B^+(u,v) = -4\mathrm{i}\omega_1\iint_{-\infty}^{\infty}\mathrm{d}u_1\,\mathrm{d}v_1\iint_{-\infty}^{\infty}\mathrm{d}u_2\,\mathrm{d}v_2\frac{\mathrm{e}^{(k_1\zeta_1+k_2\zeta_2)}\,(k_2+\nu_2)}{k_1-\nu_1^\epsilon}\,\mathrm{e}^{-\mathrm{i}(u_1\xi_1+v_1\eta_1)}$$
$$\times\,\delta(u+u_1+u_2)\,\delta(v+v_1+v_2)$$
$$= -4\mathrm{i}\omega_1\iint_{-\infty}^{\infty}\mathrm{d}u_1\,\mathrm{d}v_1\frac{\mathrm{e}^{k_1\zeta_1+\zeta_2((u+u_1)^2+(v+v_1)^2)^{\frac{1}{2}}}\,((u+u_1)^2+(v+v_1)^2)^{\frac{1}{2}}+\nu_2)}{k_1-\nu_1^\epsilon}$$
$$\times\,\mathrm{e}^{-\mathrm{i}(u_1\xi_1+\mathrm{i}v_1h_1)}. \tag{3.31}$$

In terms of the polar coordinates, (3.31) can be rewritten in the form

$$\tilde{Q}_B^+(k,\phi) = -4\mathrm{i}\omega_1\int_0^{2\pi}\mathrm{d}\psi\int_0^{\infty}\mathrm{d}l\,l\frac{[\rho(l,k,\phi-\psi)+\nu_2]}{l-\nu_1^\epsilon}\,\mathrm{e}^{\zeta_1l+\zeta_2\rho(l,k,\phi-\psi)-\mathrm{i}lR_{12}\cos(\alpha_{12}-\psi)}. \tag{3.32}$$

Equations (3.24)–(3.32) complete the derivation of the Fourier transforms of the forcing functions $Q_{A,B}^+$ for the sum-frequency problem.

3.1.2. *Difference-frequency problem*

The forcing functions corresponding to the difference-frequency problem are defined as follows

$$Q_A^-(x,y) = -\mathrm{i}(\omega_1-\omega_2)\,\nabla G_1\cdot\nabla G_2^*, \tag{3.33a}$$

$$Q_B^-(x,y) = \mathrm{i}\omega_1 G_1\frac{\partial}{\partial z}(G_{2z}^*-\nu_2G_2^*), \tag{3.33b}$$

where the * denotes the complex conjugate of the quantity involved. Their Fourier transforms are evaluated as in the sum-frequency problem. They take the form

$$\tilde{Q}_A^- = -4\mathrm{i}(\omega_1-\omega_2)\iint_{-\infty}^{\infty} \mathrm{d}u_1\,\mathrm{d}v_1 \frac{\mathrm{e}^{\zeta_1 k_1+\zeta_2((u+u_1)^2+(v+v_1)^2)^{\frac{1}{2}}}(\nu_1\nu_2+k_1^2+uu_1+vv_1)}{(k_1-\nu_1^\epsilon)\,[((u+u_1)^2+(v+v_1)^2)^{\frac{1}{2}}-(\nu_2^\epsilon)^*]}$$

$$\times\,\mathrm{e}^{-\mathrm{i}(u_1\xi_1+\mathrm{i}v_1\eta_1)}$$

$$= -4\mathrm{i}(\omega_1-\omega_2)\int_0^{2\pi}\mathrm{d}\psi\int_0^\infty \mathrm{d}l\,l\frac{[\nu_1\nu_2+l^2+kl\cos(\phi-\psi)]}{(l-\nu_1^\epsilon)\,[\rho(l,k,\phi-\psi)-(\nu_2^\epsilon)^*]}, \tag{3.34}$$

$$\tilde{Q}_B^- = -4\mathrm{i}\omega_1\iint_{-\infty}^{\infty}\mathrm{d}u_1\,\mathrm{d}v_1\frac{\mathrm{e}^{k_1\zeta_1+\zeta_2((u+u_1)^2+(v+v_1)^2)^{\frac{1}{2}}}((u+u_1)^2+(v+v_1)^2)^{\frac{1}{2}}+\nu_2)}{k_1-\nu_1^\epsilon}$$

$$\times\,\mathrm{e}^{-\mathrm{i}(u_1\xi_1+\mathrm{i}v_1\eta_1)}$$

$$= -4\mathrm{i}\omega_1\int_0^{2\pi}\mathrm{d}\psi\int_0^\infty\mathrm{d}l\,l\frac{[\rho(l,k,\phi-\psi)+\nu_2]}{l-\nu_1^\epsilon}\,\mathrm{e}^{\zeta_1 l+\zeta_2\rho(l,k,\phi-\psi)-\mathrm{i}lR_{12}\cos(\alpha_{12}-\psi)}. \tag{3.35}$$

The availability of the Fourier transforms $\tilde{Q}_{A,B}^{\pm}$ allows the definition of the respective radiation Green functions $\mathscr{R}_{A,B}^{\pm}$ by a substitution in (3.18),

$$\mathscr{R}_{A,B}^{\pm}(R,\theta,z;R_{12},\alpha_{12}) = \frac{1}{8\pi^2 g}\int_0^\infty k\,\mathrm{d}k\int_0^{2\pi}\mathrm{d}\phi\frac{\mathrm{e}^{kz-\mathrm{i}kR\cos(\phi-\theta)}}{k-N_\delta^{\pm}}\tilde{Q}_{A,B}^{\pm}(k,\phi;R_{12},\alpha_{12}), \tag{3.36}$$

where $$x = R\cos\theta,\quad y = R\sin\theta. \tag{3.37}$$

The two-dimensional form of the Fourier transforms derived so far follows from (3.24) and (3.35) for the sum- and difference-frequency problems respectively. Setting $v = v_1 = 0$ in the integrands and omitting the integration with respect to the latter variable, we obtain

$$\tilde{q}_A^{\pm}(u) = -2\pi\mathrm{i}(\omega_1\pm\omega_2)\int_{-\infty}^{\infty}\mathrm{d}u_1\frac{\mathrm{e}^{|u_1|\zeta_1+|u+u_1|\zeta_2}(\nu_1\nu_2+u_1^2+uu_1)}{(|u_1|-\nu_1^\epsilon)\,(|u+u_1|-\nu_2^{\pm\epsilon})}\,\mathrm{e}^{-\mathrm{i}u_1\xi_1}, \tag{3.38a}$$

$$\tilde{q}_B^{\pm}(u) = -2\pi\mathrm{i}\omega_1\int_{-\infty}^{\infty}\mathrm{d}u_1\frac{\mathrm{e}^{|u_1|\zeta_1+|u+u_1|\zeta_2}(|u+u_1|+\nu_2)}{|u_1|-\nu_1^\epsilon}\,\mathrm{e}^{-\mathrm{i}u_1\xi_1}. \tag{3.38b}$$

The substitution of (3.38) into (3.20) produces the two-dimensional radiation Green functions.

3.2. *The diffraction Green functions*

3.2.1. *Sum-frequency problem*

The diffraction problem is defined as the interaction of the incident wave with the source with index $i = 1$ here located under the origin of the coordinate system, so that $\xi_1 = \eta_1 = 0$ with $\zeta_1 < 0$. The corresponding forcing functions are defined by

$$Q_A^+(x,y) = -\mathrm{i}(\omega_0+\omega_1)\,(\boldsymbol{\nabla}\varphi_0\cdot\boldsymbol{\nabla}G_1)$$

$$= \frac{\omega_0(\omega_0+\omega_1)A}{\pi}\iint_{-\infty}^{\infty}\mathrm{d}u\,\mathrm{d}v\frac{u+\nu_1}{(u^2+v^2)^{\frac{1}{2}}-\nu_1^\epsilon}\,\mathrm{e}^{\zeta_1(u^2+v^2)^{\frac{1}{2}}+\mathrm{i}(u-\nu_0)x+\mathrm{i}vy}, \tag{3.39}$$

and $$Q_B^+(x,y) = \mathrm{i}\omega_0\varphi_0\frac{\partial}{\partial z}(G_{1z}-\nu_1 G_1)$$

$$= \frac{-gA}{\pi}\iint_{-\infty}^{\infty}\mathrm{d}u\,\mathrm{d}v((u^2+v^2)^{\frac{1}{2}}+\nu_1)\,\mathrm{e}^{\zeta_1(u^2+v^2)^{\frac{1}{2}}+\mathrm{i}(u-\nu_0)x+\mathrm{i}vy}. \tag{3.40}$$

The Fourier transforms of $Q^{+}_{A,B}$ are evaluated as in the radiation problem. They take the form

$$\tilde{Q}^{+}_{A}(u,v) = 4\pi\omega_0(\omega_0+\omega_1)\,A\,\frac{\nu_1+\nu_0-u}{((u-\nu_0)^2+v^2)^{\frac{1}{2}}-\nu_1^{\epsilon}}\,\mathrm{e}^{\zeta_1((u-\nu_0)^2+v^2)^{\frac{1}{2}}}, \tag{3.41}$$

$$\tilde{Q}^{+}_{B}(u,v) = -4\pi g A[((u-\nu_0)^2+v^2)^{\frac{1}{2}}+\nu_1]\,\mathrm{e}^{\zeta_1((u-\nu_0)^2+v^2)^{\frac{1}{2}}}. \tag{3.42}$$

3.2.2. *Difference-frequency problem*

Proceeding along the same lines, we define the forcing functions

$$Q^{-}_{A}(x,y) = -\mathrm{i}(\omega_0-\omega_1)\,(\nabla\varphi_0\cdot\nabla G^{*}_{1}), \tag{3.43}$$

$$Q^{-}_{B}(x,y) = \mathrm{i}\omega_0\,\varphi_0\frac{\partial}{\partial z}(G^{*}_{1z}-\nu_1 G^{*}_{1}), \tag{3.44}$$

which possess the Fourier transforms

$$\tilde{Q}^{-}_{A}(u,v) = 4\pi\omega_0(\omega_0-\omega_1)\,A\,\frac{\nu_1+\nu_0-u}{((u-\nu_0)^2+v^2)^{\frac{1}{2}}-(\nu_1^{\epsilon})^{*}}\,\mathrm{e}^{\zeta_1((u-\nu_0)^2+v^2)^{\frac{1}{2}}}, \tag{3.45}$$

$$\tilde{Q}^{-}_{B}(u,v) = -4\pi g A[((u-\nu_0)^2+v^2)^{\frac{1}{2}}+\nu_1]\,\mathrm{e}^{\zeta_1((u-\nu_0)^2+v^2)^{\frac{1}{2}}}. \tag{3.46}$$

The diffraction Green functions for the sum- and difference-frequency problems are denoted by $\mathscr{D}^{\pm}_{A,B}$ and are defined by substituting (3.41)–(3.42) and (3.45)–(3.46) into (3.18)

$$\mathscr{D}^{\pm}_{A,B}(R,\theta,z) = \frac{1}{8\pi^2 g}\int_0^{\infty} k\,\mathrm{d}k\int_0^{2\pi}\mathrm{d}\phi\,\frac{\mathrm{e}^{kz-\mathrm{i}R\cos(\phi-\theta)}}{k-N^{\pm}_{\delta}}\,\tilde{Q}^{\pm}_{A,B}(k,\phi), \tag{3.47}$$

where $\tilde{Q}^{\pm}_{A,B}(k,\phi)$ are defined by

$$\tilde{Q}^{\pm}_{A}(k,\phi) = 4\pi\omega_0(\omega_0\pm\omega_1)\,A\,\frac{\nu_1+\nu_0-k\cos\phi}{\rho(k,\nu_0,\pi+\phi)-\nu_1^{\pm\epsilon}}\,\mathrm{e}^{\zeta_1\rho(k,\nu_0,\pi+\phi)}, \tag{3.48}$$

$$\tilde{Q}^{\pm}_{B}(u,v) = -4\pi g A[\rho(k,\nu_0,\pi+\phi)+\nu_1]\,\mathrm{e}^{\zeta_1\rho(k,\nu_0,\pi+\phi)}. \tag{3.49}$$

For finite negative values of the vertical source coordinates ζ_i, both the radiation and diffraction Green functions are regular functions in the entire fluid domain. This is ensured by the exponential decay of the functions $\tilde{Q}^{\pm}_{A,B}(k,\phi)$ as the wavenumber k tends to infinity. This property of the second-order Green functions differentiates them from their linear counterpart, which is known to be $1/r$ singular near the source.

The two-dimensional form of the Fourier transforms $\tilde{Q}^{\pm}_{A,B}$ of the diffraction problem are obtained as in the radiation problem, and are defined by

$$\tilde{q}^{\pm}_{A}(u) = 2\omega_0(\omega_0\pm\omega_1)\,A\,\frac{\nu_1+\nu_0-u}{|u-\nu_0|-\nu_1^{\pm\epsilon}}\,\mathrm{e}^{\zeta_1|u-\nu_0|}, \tag{3.50}$$

$$\tilde{q}^{\pm}_{B}(u) = -2gA(|u-\nu_0|+\nu_1)\,\mathrm{e}^{\zeta_1|u-\nu_0|}. \tag{3.51}$$

The substitution of (3.50) and (3.51) into (3.20) produces the two-dimensional diffraction Green functions.

4. Formulation of the second-order problem

Consider the interaction of random ambient waves with a freely floating body. According to linear theory, a random seaway can be approximated by the linear superposition of a sufficiently large number of regular plane progressive wave

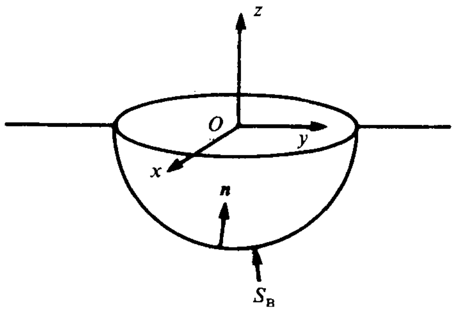

FIGURE 2

components of different frequencies and headings. The energy density of each component is defined by the ambient wave spectrum. In linear wave–body interactions, coupling between two components of the spectrum is not possible while such a coupling occurs in the second-order problem. It is therefore sufficient to group the incident-wave components in pairs, and study the second-order interaction of each pair with the body.

Figure 2 illustrates a freely floating right body and a Cartesian coordinate system fixed relative to the calm position of the free surface. Two deep-water plane progressive waves of different frequency and heading are incident upon the body. They are defined by

$$\varphi_{I1} = \frac{igA_1}{\omega_1} e^{\nu_1 z - i\nu_1 x\cos\beta_1 - i\nu_1 y \sin\beta_1 + i\delta_1}, \tag{4.1a}$$

$$\varphi_{I2} = \frac{igA_2}{\omega_2} e^{\nu_2 z - i\nu_2 x\cos\beta_2 - i\nu_2 y \sin\beta_2 + i\delta_2}, \tag{4.1b}$$

where δ_1, δ_2 are the statistically independent phases. The second-order velocity potential resulting from the interaction of the two incident waves exists and can be determined by utilizing the analysis of §3. Its sum- and difference-frequency components are defined as in (3.17), with the corresponding forcing functions given by

$$\begin{aligned} Q_A^+(x,y) &= -i\Omega^+\nabla\varphi_{I1}\cdot\nabla\varphi_{I2} \\ &= i\Omega^+ A_1 A_2 \omega_1\omega_2[1-\cos(\beta_1-\beta_2)]\,e^{-i(l^+x+m^+y)+i(\delta_1+\delta_2)}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} Q_A^-(x,y) &= -i\Omega^-\nabla\varphi_{I1}\cdot\nabla\varphi_{I2}^* \\ &= i\Omega^- A_1 A_2 \omega_1\omega_2[1+\cos(\beta_1-\beta_2)]\,e^{-i(l^-x-im^-y)+i(\delta_1-\delta_2)}, \end{aligned} \tag{4.3}$$

$$l^{\pm} = \nu_1\cos\beta_1 \pm \nu_2\cos\beta_2, \tag{4.4}$$

$$m^{\pm} = \nu_1\sin\beta_1 \pm \nu_2\sin\beta_2, \tag{4.5}$$

where $\Omega^{\pm} = \omega_1 \pm \omega_2$. The forcing functions $Q_B^{\pm}$ vanish for deep-water regular waves. Evaluating the Fourier transforms of $Q_A^{\pm}$ with respect to the (x,y)-coordinates, making use of the identity (3.23) and substituting into (3.18), we obtain the sum- and difference-frequency second-order complex velocity potentials resulting from the interaction of φ_{I1} and φ_{I2},

$$\varphi_{I12}^{\pm} = \frac{\pm i}{2}\Omega^{\pm}A_1A_2\frac{\omega_1\omega_2}{g}[1\mp\cos(\beta_1-\beta_2)]\frac{e^{z(l^{\pm 2}+m^{\pm 2})^{\frac{1}{2}}}}{(l^{\pm 2}+m^{\pm 2})^{\frac{1}{2}}-\Omega^{\pm 2}/g}\,e^{-il^{\pm}x-im^{\pm}y+i(\delta_1\pm\delta_2)}. \tag{4.6}$$

The vanishing of the denominator of (4.6) does not lead to a singularity. In the sum-frequency problem we have

$$l^{+2}+m^{+2} = \nu_1^2+\nu_2^2+2\nu_1\nu_2\cos(\beta_1-\beta_2). \tag{4.7}$$

The maximum value of this wavenumber, attained for $\beta_1 = \beta_2$, is always smaller than Ω^{+2}/g by virtue of the inequality

$$|\nu_1+\nu_2| < (\omega_1+\omega_2)^2/g. \tag{4.8}$$

The corresponding wavenumber in the difference-frequency problem is defined by

$$(l^-)^2+(m^-)^2 = \nu_1^2+\nu_2^2-2\nu_1\nu_2\cos(\beta_1-\beta_2). \tag{4.9}$$

We may assume without loss of generality that $\omega_1 > \omega_2$. The minimum value of the wavenumber (4.9), attained for $\beta_1 = \beta_2$, is always larger than Ω^{-2}/g by virtue of the inequality $\nu_1-\nu_2 > (\omega_1-\omega_2)^2/g$ which reduces to an equality when $\Omega^- = 0$. In this case, however, the singularity in the second-order potential (4.6) is offset by the multiplicative factor Ω^-. This analysis completes the solution of the second-order incident-wave potential.

The linear interaction of each incident-wave potential with a freely floating body generates a body–wave disturbance which consists of the radiation and the diffraction components. Their sum is described by the complex velocity potential φ_{B} which can be determined from the solution of a boundary-integral equation. The 'source distribution' method represents the velocity potential as follows

$$\varphi_{\mathrm{B}}(\boldsymbol{x}) = \iint_{S_b} \mathrm{d}\xi\, \sigma(\xi)\, G(\boldsymbol{x};\xi). \tag{4.10}$$

The Green function $G(\boldsymbol{x};\xi)$ is defined by (2.13) and the source strength is obtained from the solution of the Fredholm integral equation

$$2\pi\sigma(\boldsymbol{x})+\iint_{S_b} \mathrm{d}\xi\, \sigma(\xi)\frac{\partial G(\boldsymbol{x};\xi)}{\partial n_x} = \frac{\partial \varphi_{\mathrm{B}}(\boldsymbol{x})}{\partial n_x} = V^{(1)}(\boldsymbol{x}), \tag{4.11}$$

where the function $V^{(1)}(\boldsymbol{x})$ is the known normal velocity of the body boundary. An alternative popular formulation is based on the application of Green's theorem and leads to a similar integral equation for the velocity potential. It takes the form

$$2\pi\varphi_{\mathrm{B}}(\boldsymbol{x})+\iint_{S_b} \mathrm{d}\xi\, \varphi_{\mathrm{B}}(\xi)\frac{\partial G(\xi;\boldsymbol{x})}{\partial n_\xi} = \iint_{S_b} \mathrm{d}\xi\, \frac{\partial \varphi_{\mathrm{B}}(\xi)}{\partial n_\xi} G(\xi;\boldsymbol{x}). \tag{4.12}$$

Following the solution of (4.12) for $\varphi_{\mathrm{B}}(\boldsymbol{x})$ on the body boundary, the same equation can be used to represent the velocity potential in the fluid domain, with the leading factor 2π replaced by 4π.

Denote by $\varphi_{\mathrm{B}i}, i = 1,2$ the linear wave disturbances resulting from the interaction of each of the two incident waves with the body. The total linear disturbance is defined by the velocity potential

$$\Phi^{(1)} = \mathrm{Re}\,[(\varphi_{\mathrm{I}1}+\varphi_{\mathrm{B}1})\,\mathrm{e}^{\mathrm{i}\omega_1 t}+(\varphi_{\mathrm{I}2}+\varphi_{\mathrm{B}2})\,\mathrm{e}^{\mathrm{i}\omega_2 t}]. \tag{4.13}$$

The second-order problem accounts for interactions of the four components contained in (4.13), complying to the second-order free-surface and body-boundary conditions. The former is determined by substituting (4.13) into (3.2). The latter is enforced on the mean position of the body boundary and is derived by Ogilvie (1983) in the form

$$\frac{\partial \Phi^{(2)}}{\partial n} = V^{(2)}(\boldsymbol{x}), \tag{4.14a}$$

$$V^{(2)}(\boldsymbol{x}) = \boldsymbol{n}\cdot\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\xi}^{(2)}(t)+\boldsymbol{\alpha}^{(2)}(t)\times\boldsymbol{x}+\boldsymbol{H}(t)\,\boldsymbol{x})-\boldsymbol{n}\cdot[(\boldsymbol{\xi}^{(1)}+\boldsymbol{\alpha}^{(1)}\times\boldsymbol{x})\cdot\boldsymbol{\nabla}]\,\boldsymbol{\nabla}\Phi^{(1)}$$
$$+(\boldsymbol{\alpha}^{(1)}\times\boldsymbol{n})\cdot\left[\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\xi}^{(1)}(t)+\boldsymbol{\alpha}^{(1)}(t)\times\boldsymbol{x})-\boldsymbol{\nabla}\Phi^{(1)}\right], \tag{4.14b}$$

where $\boldsymbol{\xi}^{(1)}, \boldsymbol{\alpha}^{(1)}$ and $\boldsymbol{\xi}^{(2)}, \boldsymbol{\alpha}^{(2)}$ are the linear and second-order translation and rotation vectors. The components of the former are given by $\boldsymbol{\xi}^{(1)} = (\xi_1, \xi_2, \xi_3)$, $\boldsymbol{\alpha}^{(1)} = (\xi_4, \xi_5, \xi_6)$ and the matrix $\boldsymbol{H}$ is defined by

$$\boldsymbol{H} = -\frac{1}{2}\begin{pmatrix} \xi_5^2+\xi_6^2 & 0 & 0 \\ -2\xi_4\,\xi_5 & \xi_4^2+\xi_6^2 & 0 \\ -2\xi_4\,\xi_6 & -2\xi_5\,\xi_6 & \xi_4^2+\xi_5^2 \end{pmatrix}. \tag{4.14c}$$

Since the boundary-value problem governing the second-order potential is linear, the second-order disturbance can be obtained by the linear superposition of two components. The first satisfies the non-homogeneous body-boundary condition (4.14) and the linear free-surface condition, and can be determined from the solution of an integral equation similar to (4.11) or (4.12). Equation (4.14) requires the evaluation of second derivatives of the linear velocity potential on the body boundary, which may not be easy to obtain with sufficient accuracy by existing methods of solution of the linear problem. It will be shown in §7 that if only the integrated forces are desired, they can be obtained from a reciprocity relation which requires the evaluation only of single derivatives of the linear potential on the body boundary. The second component of the second-order problem satisfies the free-surface condition (3.1)–(3.2) and a homogeneous Neumman condition on the mean position on the body boundary. The determination of this latter disturbance will be the focus of the remainder of this and the next two sections.

Following the convention adopted in §3, the word radiation labels the second-order disturbance driven by quadratic products of linear body–wave disturbances, including the linear diffraction potential. This use is justified by the similar properties of the linear radiation and diffraction velocity potentials. The word diffraction is reserved for the second-order component driven by quadratic products of the incident and body–wave disturbances.

Two characteristic bichromatic 'radiation' and two 'diffraction' forcing terms are sufficient to consider in the free-surface condition (3.2). In the radiation problem they are defined by

$$Q_A^{\mathrm{R}}(x,y,t) = -\frac{\partial}{\partial t}(\boldsymbol{\nabla}\Phi_{\mathrm{B}1}\cdot\boldsymbol{\nabla}\Phi_{\mathrm{B}2}), \tag{4.15}$$

$$Q_B^{\mathrm{R}}(x,y,t) = \frac{1}{g}\Phi_{\mathrm{B}1t}\frac{\partial}{\partial z}(\Phi_{\mathrm{B}2tt}+g\Phi_{\mathrm{B}2z}), \tag{4.16}$$

and in the diffraction problem by

$$Q_A^{\mathrm{D}}(x,y,t) = -\frac{\partial}{\partial t}(\boldsymbol{\nabla}\Phi_{\mathrm{I}1}\cdot\boldsymbol{\nabla}\Phi_{\mathrm{B}2}), \tag{4.17}$$

$$Q_B^{\mathrm{D}}(x,y,t) = \frac{1}{g}\Phi_{\mathrm{I}1t}\frac{\partial}{\partial z}(\Phi_{\mathrm{B}2tt}+g\Phi_{\mathrm{B}2z}). \tag{4.18}$$

The bichromatic character of these forcing terms permits the construction of the full second-order solution driven by the linear velocity potential (4.13) as the linear superposition of the radiation and diffraction solutions obtained from the forcing terms (4.15)–(4.18). Where necessary the frequency of linear disturbance with index $i = 1$ may be allowed to coalesce with the frequency of the disturbance with index $i = 2$.

We conclude the present section with the statement of the typical boundary-value problem governing the second-order potential

$$\nabla^2\varphi = 0, \tag{4.19}$$

$$-\Omega^2\varphi + g\varphi_z = Q(x, y), \quad \text{on } z = 0, \tag{4.20}$$

$$\frac{\partial\varphi}{\partial n} = 0, \quad \text{on } S_b. \tag{4.21}$$

The set of equations (4.19)–(4.21) must be supplemented by a radiation condition in order to ensure uniqueness. Direct boundary-integral numerical solutions of (4.19)–(4.21) which discretize the free surface and utilize the Rankine source $1/r$ or the linear Green function (2.13) as the elementary solution, must make explicit use of this radiation condition over the infinite end of their computational domain. The explicit use of a radiation condition is unnecessary if natural elementary solutions or Green functions are used to solve the associated boundary-value problem. The linear wave source potential (2.13) is, for example, the natural elementary solution of the linear problem invoking the Sommerfeld radiation condition implicitly via its definition. The natural elementary solutions for the boundary-value problem (4.19)–(4.21) forced by the four functions (4.15)–(4.18) are the four Green functions derived in §3. As is the case with the linear Green function, radiation conditions are contained in their definitions, but will not be explicitly used for the solution of the second-order problem presented in the next section.

5. Solution of the second-order problem

The method of solution for the sum- and difference-frequency problems, and for the components in each problem driven by forcing terms of the A- of B-type is the same. Any distinction between them is not essential in the ensuing derivation and is dropped. We adopt the convention that the radiation Green function $\mathscr{R}$ corresponds to any of its four components defined in (3.36). A similar convention is adopted for the diffraction Green function $\mathscr{D}$ (see equation (3.47)). The radiation and diffraction problems will be considered separately, with the latter treated first due to its relative algebraic simplicity.

5.1. *Diffraction problem*

The diffraction velocity potential $\varphi(\boldsymbol{x})$ is subject to the body-boundary and free-surface conditions

$$\frac{\partial\varphi}{\partial n} = 0, \quad \text{on } S_b, \tag{5.1}$$

$$-\Omega^2\varphi + g\varphi_z = Q^{\mathrm{D}}(x, y), \quad \text{on } z = 0, \tag{5.2}$$

where the function $Q^{\mathrm{D}}(x, y)$ is the sum- or difference-frequency component of the forcing terms defined by (4.17) or (4.18). In connection with the source distribution method, the linear body-wave potential φ_{B2} is defined by

$$\varphi_{\mathrm{B2}}(\boldsymbol{x}) = \iint_{S_b} \mathrm{d}\xi\, \sigma_2(\xi)\, G_2(\boldsymbol{x}; \xi). \tag{5.3}$$

Define the velocity potential

$$\varphi_{\mathrm{P}}(\boldsymbol{x}) = \iint_{S_b} \mathrm{d}\xi\, \sigma_2(\xi)\, \mathscr{D}(\boldsymbol{x}; \xi), \tag{5.4}$$

where the diffraction Green function $\mathscr{D}$ is subject to either of the free-surface conditions (§3)

$$-\Omega^2 \mathscr{D}(\boldsymbol{x}; \xi) + g\mathscr{D}_z(\boldsymbol{x}; \xi) = -\mathrm{i}\Omega \nabla\varphi_{\mathrm{I}1} \cdot \nabla_x G_2(\boldsymbol{x}; \xi), \tag{5.5a}$$

or

$$-\Omega^2 \mathscr{D}(\boldsymbol{x}; \xi) + g\mathscr{D}_z(\boldsymbol{x}; \xi) = -\mathrm{i}\omega_1 \varphi_{\mathrm{I}1} \frac{\partial}{\partial z} (G_{2z}(\boldsymbol{x}; \xi) - \nu_2 G_2(\boldsymbol{x}; \xi)), \tag{5.5b}$$

with $\varphi_{\mathrm{I}1}$ defined by (4.1 a) and G_2 by (2.13 b). The right-hand side of the free-surface conditions (5.5) here corresponds to the sum-frequency problem. The ensuing derivation can be repeated in an identical manner for the difference-frequency problem.

Taking the inner product of the gradient of (5.3) with $\nabla\varphi_{\mathrm{I}1}$, interchanging the differentiation and integration and using the free-surface condition (5.5), we obtain

$$\begin{aligned} Q^{\mathrm{D}}(x, y) &= -\mathrm{i}\Omega \nabla\varphi_{\mathrm{I}1}(\boldsymbol{x}) \cdot \nabla\varphi_{\mathrm{B2}}(\boldsymbol{x}) \\ &= \iint_{S_b} \mathrm{d}\xi\, \sigma_2(\xi)\, [-\mathrm{i}\Omega \nabla\varphi_{\mathrm{I}1}(\boldsymbol{x}) \cdot \nabla_x G_2(\boldsymbol{x}; \xi)] \\ &= \iint_{S_b} \mathrm{d}\xi\, \sigma_2(\xi) \left(-\Omega^2 + g\frac{\partial}{\partial z}\right) \mathscr{D}(\boldsymbol{x}; \xi) \\ &= -\Omega^2 \varphi_{\mathrm{P}} + g\varphi_{\mathrm{P}_z}. \end{aligned} \tag{5.6}$$

Equation (5.6) shows that the velocity potential $\varphi_{\mathrm{P}}(\boldsymbol{x})$ satisfies the non-homogeneous free-surface condition (4.17)–(5.5 a). A similar proof holds for the free-surface condition (4.18)–(5.5 b), as long as the relevant Green function is used. The interchange of the differentiation and integration in (5.6) is permissible for $\boldsymbol{x} \neq \xi$ which is the case when $\boldsymbol{x}$ is located on the free surface and ξ on the body boundary. Care must be exercized near the intersection of the body with the free surface. The local behaviour of the second-order solution will be analysed later in this section.

The velocity potential $\varphi_{\mathrm{P}}(\boldsymbol{x})$ is hereinafter referred to as the particular solution because it satisfies the non-homogeneous free-surface condition. The body boundary condition is enforced by the addition of a homogeneous solution $\varphi_{\mathrm{H}}(\boldsymbol{x})$, subject to the linear boundary-value problem

$$-\Omega^2 \varphi_{\mathrm{H}} + g\varphi_{\mathrm{H}_z} = 0, \quad \text{on } z = 0, \tag{5.7}$$

$$\frac{\partial\varphi_{\mathrm{H}}}{\partial n} = -\frac{\partial\varphi_{\mathrm{P}}}{\partial n} \quad \text{on } S_b. \tag{5.8}$$

Therefore the solution of the second-order problem is

$$\varphi = \varphi_{\mathrm{P}} + \varphi_{\mathrm{H}}, \tag{5.9}$$

where φ_{P} is determined explicitly in terms of the diffraction Green functions and the linear source strength distribution $\sigma_2(\boldsymbol{x})$ (equation (5.4)), and φ_{H} is obtained from the solution of the linear problem (5.7)–(5.8). In taking the normal derivative of the particular solution in (5.8), no mention was made of its behaviour in the vicinity of the body boundary. The velocity potential $\varphi_{\mathrm{P}}(\boldsymbol{x})$ is harmonic in the entire fluid domain, including the domain interior to the body boundary. This follows from the regularity of the second-order Green functions (3.47)–(3.49) at $\boldsymbol{x} = \boldsymbol{\xi}$.

In connection with the Green method, the particular solution φ_{P} is defined by

$$\varphi_{\mathrm{P}}(\boldsymbol{x}) = \frac{1}{4\pi}\iint_{S_b} \mathrm{d}\xi \left(\frac{\partial\varphi_{\mathrm{B2}}(\xi)}{\partial n_\xi} - \varphi_{\mathrm{B2}}(\xi)\frac{\partial}{\partial n_\xi}\right)\mathscr{D}(\boldsymbol{x};\xi). \tag{5.10}$$

The remainder of the solution is the same.

5.2. *Radiation problem*

The radiation velocity potential is again denoted by φ and is subject to the conditions

$$\frac{\partial\varphi}{\partial n} = 0, \quad \text{on } S_b, \tag{5.11}$$

$$-\Omega^2\varphi + g\varphi_z = Q^{\mathrm{R}}(x,y), \quad \text{on } z = 0, \tag{5.12}$$

where the forcing function in (5.12) is the sum- or difference-frequency component of the forcing functions defined by (4.15) or (4.16). Since the radiation Green functions represent the second-order interaction between two distinct submerged sources, the particular solution for the radiation problem takes the form

$$\varphi_{\mathrm{P}}(\boldsymbol{x}) = \iint_{S_b} \mathrm{d}\xi_1\,\sigma_1(\xi_1)\iint_{S_b}\mathrm{d}\xi_2\,\sigma_2(\xi_2)\,\mathscr{R}(\boldsymbol{x};\xi_1,\xi_2). \tag{5.13}$$

The proof that this particular solution satisfies the free-surface conditions (4.15) or (4.16), when equipped with the radiation Green function of the A- or B-type respectively, follows the steps of the corresponding proof in the diffraction problem. This particular solution is again supplemented by a homogeneous component φ_{H} subject to the equations (5.7)–(5.8). The regularity of the particular solution at the interior of the body boundary is valid in the radiation problem as well, by virtue of the regularity of the corresponding Green functions defined by (3.36).

In connection with the Green method, the particular solution of the radiation problem can be expressed in the form

$$\begin{aligned}\varphi_{\mathrm{P}}(\boldsymbol{x}) = \frac{1}{4\pi}\iint_{S_b}\mathrm{d}\xi_1\left(\frac{\partial\varphi_{\mathrm{B1}}(\xi_1)}{\partial n_{\xi_1}} - \varphi_{\mathrm{B1}}(\xi_1)\frac{\partial}{\partial n_{\xi_1}}\right) \\ \times\iint_{S_b}\mathrm{d}\xi_2\left(\frac{\partial\varphi_{\mathrm{B2}}(\xi_2)}{\partial n_{\xi_2}} - \varphi_{\mathrm{B2}}(\xi_1)\frac{\partial}{\partial n_{\xi_2}}\right)\mathscr{R}(\boldsymbol{x};\xi_1,\xi_2).\end{aligned} \tag{5.14}$$

It follows from the present method of solution of the radiation and diffraction problems that the properties of the second-order velocity potential are shared by the particular and homogeneous components. The former, being regular in the entire fluid domain, can be regarded as the 'incident flow' forced by the non-homogeneous free-surface condition. Its behaviour at infinity can be determined by the far-field asymptotics of the second-order Green functions. The homogeneous component can be regarded as the 'disturbance flow' which no longer needs to satisfy the non-homogeneous free-surface condition, a 'burden' assumed by the particular component. Thus, it is determined from the solution of a linear problem and is subject to the Sommerfeld radiation condition at infinity. The sum of the particular and homogeneous component is 'a' solution of the boundary-value problem (4.19)–(4.21), satisfying the radiation condition at infinity implied by the explicit definition of the former and the method of solution of the latter. To establish that this is 'the only' solution of this problem we need a uniqueness theorem which we lack in the second-order problem. A proof that solutions exist is often harder to establish. This is, however, not the case with the present method of solution. The explicit form of the particular component and the availability of existence theorems in the linear problem (John 1950) governing the homogeneous component, are sufficient to establish the existence of a solution for a certain class of body geometries.

5.3. *Solution behaviour near the intersection of body with free surface*

In the vicinity of the body waterline the particular solution develops a singular behaviour which originates from the singularity of the second-order Green functions when $\zeta_i = 0$ and (x, y, z) approaches their location on the free surface. A similar singular behaviour is present in the homogeneous solution φ_{H}, related to φ_{P} by the boundary-value problem (5.7)–(5.8). The structure of this singular behaviour will be studied next. Aside from its fundamental interest, its knowledge is essential for the development of a robust numerical solution of the second-order problem for surface-piercing bodies.

Over radial distances from the body waterline small compared to the local radius of curvature, the flow may be assumed to be locally two-dimensional. We will therefore confine our attention to the two-dimensional second-order flow in the vicinity of the intersection of a body section with the free-surface. While convenient from the algebraic standpoint, this reduction is not expected to be restrictive in revealing the nature of the singular behaviour near the waterline in the three-dimensional problem. Only bodies which pierce the free surface at right-angles will be considered. This assumption is not restrictive for a wide range of bodies encountered in practice, it leads to a more regular behavour relative to the case where the intersection angle differs from 90°, and is less testing of the perturbation expansion in the vicinity of the body waterline.

The non-homogeneous terms in the second-order free-surface condition are functions of a linear solution the local behaviour of which will be studied first. The linear velocity $\phi(\boldsymbol{x})$ is subject to the free-surface and wall conditions

$$\phi_z - \nu\phi = 0 \quad \text{on } z = 0, \tag{5.15}$$

$$\phi_x = U(z) \quad \text{on } x = 0. \tag{5.16}$$

The wall is assumed to be vertical over a finite distance beneath the free surface. The ensuing analysis will isolate and determine the 'non-analytic' component of ϕ near $r = (x^2 + z^2)^{\frac{1}{2}} = 0$, omitting contributions which are analytic at the origin of the coordinate system.

The 'reduced potential'

$$\chi(x, z) = \phi_z - \nu\phi, \tag{5.17}$$

is subject to

$$\chi(x, 0) = 0, \tag{5.18}$$

and

$$\chi_x(0, z) = U'(z) - \nu U(z) = V(z). \tag{5.19}$$

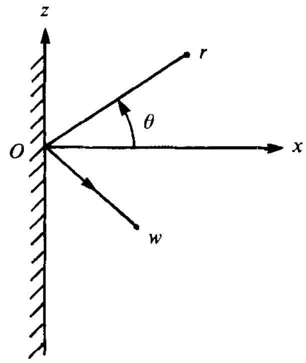

FIGURE 3

The limiting value $V_0 = V(z=0)$ will be hereinafter used in the boundary conditions (5.19) for the derivation of the leading-order behaviour of the velocity potentials χ and ϕ as $r = (x^2+z^2)^{\frac{1}{2}} \rightarrow 0$. Higher-order terms in the Taylor series expansion of $V(z)$ as $z \rightarrow 0$ vanish at the origin and contribute higher-order corrections. In terms of the complex variable $w = x + \mathrm{i}z = r\,\mathrm{e}^{\mathrm{i}\theta}$, the solution of (5.18)–(5.19) for the complex reduced potential $G(w) = \chi + \mathrm{i}\tau$ is

$$G(w) = \frac{V_0}{\pi}(-\mathrm{i}w \log w). \tag{5.20}$$

The complex velocity potential $F(w) = \phi(x,z) + \mathrm{i}\psi(x,z)$ is related to the complex reduced potential $G(w)$ by the relation

$$G(w) = \mathrm{i}F'(w) - \nu F(w), \tag{5.21}$$

which follows from the definitions (5.17). The solution of (5.21) for $F(w)$ is

$$F(w) = -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\nu w} \int_{w_0}^{w} \mathrm{d}v\, \mathrm{e}^{\mathrm{i}\nu v} G(v), \tag{5.22}$$

where w_0 is a fixed complex number. Different selections for w_0 lead to differences in $F(w)$ which are analytic functions of the complex variable w not affecting its singular structure as $w \rightarrow 0$. Here we select $u_0 = 0$, along with a contour of integration which is the radius connecting the origin with the complex location w (see figure 3). Over this contour, $\mathrm{d}w = \mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}r$, and the modulus of the dummy variable v is bounded by the modulus of w. Keeping the leading-order term in the Taylor series expansion of the exponential terms in (5.22) and substituting the solution (5.20) for the complex reduced potential, we obtain for the real linear potential

$$\phi(r,\theta) \sim \frac{V_0}{\pi} r^2(-\cos 2\theta \log r + \theta \sin 2\theta). \tag{5.23}$$

Certain properties of (5.23) deserve some discussion. For a body oscillating in heave, $V(z) = 0$, leading to a linear heave velocity potential which is analytic at $r = 0$. This analytic behaviour is also shared by the diffraction velocity potential, for which $U(z) \propto \mathrm{e}^{\nu z}$, or $V(z) = 0$. For a sway oscillation, $V_0 = -\nu U$. In this case the local singular behaviour of the sway velocity potential decreases in magnitude with decreasing wavenumber ν, leading to an analytical behaviour in the zero-frequency limit. For a rotational oscillation around the origin O with an angular velocity α,

$V_0 = \alpha$, suggesting that the singular behaviour persists at all frequencies. Moreover, the r-dependence of the expansion (5.23) suggests that the singular component of the linear solution does not contribute to the body boundary condition at the intersection point, since $(1/r)(\partial\phi/\partial\theta) \to 0$ on $\theta = -\frac{1}{2}\pi$ as $r \to 0$ when ϕ is supplied by (5.23). It follows that the body boundary condition is locally enforced by the analytic component of ϕ which is omitted in (5.23). This analysis of the linear solution confirms that the value of the velocity potential and its first spatial derivatives are bounded at the intersection of the body section with the free surface. A more singular behaviour is expected if the intersection angle differs from 90°. A detailed discussion of the local behaviour of the flow and review of earlier studies is presented by Stoker (1957).

Turning to the second-order potential, we define two local second-order problems for the real velocity potentials $\varphi_{A,B}$:

$$\mathscr{X}_A = \frac{\partial\varphi_A}{\partial z} - N\varphi_A = \nabla\phi_1 \cdot \nabla\phi_2, \quad \text{on } z = 0, \tag{5.24}$$

$$\frac{\partial\varphi_A}{\partial x} = \frac{\partial\mathscr{X}_A}{\partial x} = 0, \quad \text{on } z = 0, \tag{5.25}$$

and

$$\mathscr{X}_B = \frac{\partial\varphi_B}{\partial z} - N\varphi_B = \phi_1 \frac{\partial}{\partial z}(\phi_{2z} - \nu\phi_2) \quad \text{on } z = 0, \tag{5.26}$$

$$\frac{\partial\varphi_B}{\partial x} = \frac{\partial\mathscr{X}_B}{\partial x} = 0, \quad \text{on } x = 0, \tag{5.27}$$

where N is either the sum- or the difference-frequency wavenumber.

Consider first the solution of problem (5.24)–(5.25). Using the expansion (5.23) of the linear solution on $\theta = 0$, the leading-order behaviour of the free-surface condition (5.24) as $x \to 0$ is

$$\mathscr{X}_A = C_A\, x \log|x| + O(x^2 \log|x|), \quad \text{on } z = 0, \tag{5.28}$$

where C_A is a multiplicative constant, function of the reduced velocities of the linear velocity potentials ϕ_1 and ϕ_2. The complex velocity potential $\mathscr{G}_A(w) = \mathscr{X}_A(r,\theta) + \mathrm{i}\mathscr{T}_A(r,\theta)$ with real part obeying the boundary conditions (5.28) and (5.25), is given by

$$\mathscr{G}_A(w) = -\frac{C_A}{\pi} \mathrm{i}w \log^2(\mathrm{i}w). \tag{5.29}$$

The solution for the corresponding complex velocity potential $\mathscr{F}(w) = \varphi_A(r,\theta) + \mathrm{i}\psi_A(r,\theta)$ is obtained by substituting (5.29) into (5.22) and proceeding as in the linear problem. After some simple algebra, we obtain for the second-order potential φ_A

$$\varphi_A(r,\theta) \sim -\frac{C_A}{\pi}\frac{r^2}{2}[\cos 2\theta(\log^2 r - \log r + \tfrac{1}{2}) - 2(\log r - \tfrac{1}{2})(\theta + \tfrac{1}{2}\pi)\sin 2\theta + (\theta + \tfrac{1}{2}\pi)^2 \cos 2\theta]. \tag{5.30}$$

The leading-order singular behaviour (5.28) of the free surface condition is the same if any of the linear velocity potentials $\phi_{1,2}$ is replaced by the incident-wave potential. Thus, the expansion (5.30) characterizes the behaviour near the intersection of the body and the free surface of both the second-order radiation and diffraction velocity potentials forced by the non-homogeneous free-surface condition (5.24).

The leading-order behaviour as $x \to 0$ of the free-surface condition (5.26) is again obtained by using (5.23) on $\theta = 0$. It follows that

$$\mathscr{X}_B = C_B \log|x| + O(x \log|x|), \quad \text{on } z = 0. \tag{5.31}$$

The complex reduced velocity potential subject to (5.31) on the free surface and (5.27) on the wall is simply given by

$$\mathscr{G}_B(w) = C_B \log w, \tag{5.32}$$

and upon substitution in (5.22), it follows that

$$\varphi_B(r, \theta) \sim C_B\, r(\sin\theta \log r + \theta \cos\theta). \tag{5.33}$$

Equation (5.33) indicates that the component of the second-order solution forced by the free-surface condition (5.26) is more singular than the component forced by condition (5.24), possessing a finite value but an unbounded vertical derivative at the intersection. It is here important to emphasize that expansions (5.30) and (5.33) characterize the singularity of the second-order solution when the linear sway or roll radiation potentials are involved in the second-order free-surface conditions (5.24) and (5.26). The linear heave radiation and diffraction potentials are analytic at the intersection for a 90° intersection angle, and their presence in the second-order free-surface condition generates an analytic local behaviour for the corresponding second-order potential.

In existing numerical solutions of the second-order problem, integrals are being evaluated by quadrature over the free surface involving in their integrand the non-homogeneous terms of the free-surface condition. It follows from the limiting behaviour (5.31) that in the two-dimensional problem these terms are logarithmically singular. A similar singular behaviour is expected in three dimensions and must be carefully accounted for in an integration by quadrature.

The singular behaviour (5.30) and (5.33) of the two components in the second-order velocity potential are valid independently of the method of solution of the boundary-value problem (4.19)–(4.21). The method of solution presented in the first part of the present section constructs the second-order potential as the superposition of a particular and a homogeneous component. It will next be shown that either is no more singular than their sum. Knowledge of the singular behaviour of the homogeneous solution will be necessary in the last part of this section for the derivation of reciprocity relations for the second-order forces and moments acting on the body. It is sufficient to establish the singular structure of the particular solution near the intersection. The homogeneous solution φ_{H} can be no more singular that the particular solution φ_{P} by virtue of the body boundary condition (5.8) relating them. The particular solution is subject to a non-homogeneous free-surface condition obtained by using the linear solution φ_{B2} defined by (5.3). For a wall-sided body section, the use of the two-dimensional form of the Green function G_2 in (5.3) leads to the normal velocity $U(z) = \frac{1}{2}\sigma(z)$, satisfied by φ_{B2} on the body boundary in the vicinity of the intersection point. Therefore, the local singular behaviour of this linear velocity potential φ_{B2} is described by the expansion (5.23).

As is the case for the total second-order potential, the local singularity structure of the radiation and diffraction particular solutions is the same, thus only the latter will be studied here. The diffraction particular solution is defined by (5.4). Because of the regularity of the Green function $\mathscr{D}(\boldsymbol{x}, \boldsymbol{\xi})$ this representation allows its definition in the entire vicinity of the intersection point including the interior of the body boundary, while enforcing no condition on its boundary. The derivation of the local singular behaviour of φ_{P} will make use of the local expansion of the free-surface condition both for $x > 0$ and $x < 0$ on $z = 0$. Starting with the free-surface condition of the A-type (equation (5.24)) and using the expansion (5.23) of the linear potential on $\theta = 0$ and $\theta = \pi$, we obtain for the corresponding reduced potential

$$\mathscr{X}_A^{\mathrm{P}}(x, 0) \sim C_A^{\mathrm{P}} x \log|x|. \tag{5.34}$$

Condition (5.34), valid on $z = 0$ both for positive and negative values of x, is sufficient to determine locally the complex reduced potential $\mathscr{G}_A^{\mathrm{P}}(w)$

$$\mathscr{G}_A^{\mathrm{P}}(w) \sim C_A^{\mathrm{P}} w \log w, \tag{5.35}$$

and upon substitution in (5.22), the velocity potential $\varphi_A^{\mathrm{P}}(r, \theta)$ is obtained in the form

$$\varphi_A^{\mathrm{P}}(r, \theta) \sim C_A^{\mathrm{P}} r^2(\sin 2\theta \log r + \theta \cos 2\theta). \tag{5.36}$$

The expansion (5.36) confirms that the A-component of the particular solution is no more singular than the corresponding component of the total second-order potential (equation (5.30)). The corresponding behaviour of the homogeneous component follows from the definition (5.9) and the expansions (5.30) and (5.36) and is no more singular than either the total or the particular solutions. Of interest in the development of a panel method for the numerical solution of the second-order problem is the behaviour of the normal velocity of the particular solution on $x = 0$ as $z \rightarrow 0$. It follows by a differentiation of (5.36) that its x-derivative on $x = 0$ is proportional to $z \log|z|$. By virtue of (5.8), a similar behaviour of opposite sign is shared by the derivative of the homogeneous component φ_A^{H}.

For the B-component of the particular solution, the local behaviour of the condition satisfied by the corresponding reduced potential on the free surface is

$$\mathscr{X}_B^{\mathrm{P}}(x, 0) \sim C_B^{\mathrm{P}} \log|x|, \tag{5.37}$$

leading to the complex reduced potential

$$\mathscr{G}_B^{\mathrm{P}}(w) \sim C_B^{\mathrm{P}} \log w, \tag{5.38}$$

and to the real velocity potential

$$\varphi_B^{\mathrm{P}}(r, \theta) \sim C_B^{\mathrm{P}} r(\sin\theta \log r + \theta \cos\theta). \tag{5.39}$$

It follows from (5.38) that the singular behaviour of the B-component of the particular solution is symmetric relative to the $x = 0$ axis thus its normal velocity on the wall vanishes. This property of the leading-order normal velocity of the B-component of the particular solution is naturally shared by the normal velocity of homogeneous solution defined by (5.8), and suggests that the B-component of the homogeneous solution is less singular near the intersection than its particular counterpart.

6. The second-order forces and moments

The second-order hydrodynamic pressure force obtained from the solution of the radiation and diffraction problems (5.1)–(5.2) and (5.11)–(5.12) respectively, is obtained from the linear Bernoulli equation

$$p = -\mathrm{i}\Omega\rho(\varphi_{\mathrm{P}} + \varphi_{\mathrm{H}}). \tag{6.1}$$

The corresponding second-order forces and moments on the body are obtained by the integration of (6.1) over the mean position of its boundary

$$X_i = -\mathrm{i}\Omega\rho \iint_{S_b} (\varphi_{\mathrm{P}}+\varphi_{\mathrm{H}})\, n_i \,\mathrm{d}s \quad (i = 1, \ldots, 6), \tag{6.2}$$

where $\boldsymbol{n} = (n_1, n_2, n_3)$ and $(n_4, n_5, n_6) = (x, y, z) \times \boldsymbol{n}$. An alternative form of the exciting-force expression (moments are understood hereafter) can be obtained by the introduction of an auxiliary linear velocity potential $\psi_i(\boldsymbol{x})$ subject to

$$-\Omega^2\psi_i + g\psi_{iz} = 0, \quad \text{on } z = 0, \tag{6.3}$$

$$\frac{\partial \psi_i}{\partial n} = n_i, \quad \text{on } S_b. \tag{6.4}$$

Both φ_{H} and ψ_i satisfy the linear free-surface condition and a condition of outgoing waves at infinity. Applying Green's identity, it follows that

$$\iint_{S_b} \left(\varphi_{\mathrm{H}} \frac{\partial \psi_i}{\partial n} - \psi_i \frac{\partial \varphi_{\mathrm{H}}}{\partial n}\right) \mathrm{d}s = 0. \tag{6.5}$$

Making use of the body-boundary condition (5.8), we may cast (6.5) in the form of the Haskind relations for the linear exciting forces, or

$$X_i = -\mathrm{i}\Omega\rho \iint_{S_b} \left(n_i \varphi_{\mathrm{P}} - \psi_i \frac{\partial \varphi_{\mathrm{P}}}{\partial n}\right) \mathrm{d}s. \tag{6.6}$$

The derivation of (6.6) using (6.5) made use of the singular behaviour of the homogeneous solution φ_{H} near the intersection to verify that there exists no localized contribution to Green's identity coming from the part of the surface of integration in (6.5) enclosing the vicinity of the body waterline. Expression (6.6) circumvents the solution for the homogeneous component φ_{H}, replacing it by the auxiliary velocity potential ψ_i.

Denote by φ_{B} the sum- or difference-frequency second-order velocity potential subject to the body boundary condition (4.14) and the linear free-surface condition. Being the solution of a linear problem, the velocity potential φ_{B} satisfies the Sommerfeld radiation condition at infinity and the corresponding second-order force can be obtained either by direct pressure integration

$$X_i^{\mathrm{B}} = -\mathrm{i}\Omega\rho \iint_{S_b} \varphi_{\mathrm{B}}\, n_i \,\mathrm{d}s \quad (i = 1, \ldots, 6), \tag{6.7}$$

or by the reciprocity relation

$$X_i^{\mathrm{B}} = -\mathrm{i}\Omega\rho \iint_{S_b} \psi_i \frac{\partial \varphi_{\mathrm{B}}}{\partial n} \mathrm{d}s, \tag{6.8}$$

with the auxiliary velocity potential ψ_i defined by (6.3)–(6.4). The derivation of (6.8) from (6.7) made use of Green's identity (6.5), where again no localized contribution from the vicinity of the body waterline arises. The use of (6.8) versus (6.7) entails no savings in computational effort since the determination of φ_{B} in (6.7) is replaced by the determination of ψ_i in (6.8). In the former case, however, double derivatives of the linear velocity potential must be evaluated on the body boundary in order to enforce the condition (4.14). It will next be shown that their determination can be avoided if (6.8) is used. We hereinafter confine our attention to the component of φ_B subject to the normal velocity

$$\frac{\partial \varphi_B}{\partial n} = \boldsymbol{n}\cdot[\boldsymbol{b}(\boldsymbol{x})\cdot\boldsymbol{\nabla}]\,\boldsymbol{\nabla}\phi \tag{6.9}$$

on the body boundary, where

$$\boldsymbol{b}(\boldsymbol{x}) = (b_1, b_2, b_3) = (\xi_1+z\xi_5-y\xi_6)\,\boldsymbol{i}+(\xi_2-z\xi_4+x\xi_6)\boldsymbol{j}+(\xi_3+y\xi_4-x\xi_5)\,\boldsymbol{k} \tag{6.10}$$

is the complex amplitude of the linear oscillatory body displacement at location $\boldsymbol{x}$ of its boundary, and ϕ is the complex linear potential. The form of (6.9) corresponds to the sum-frequency problem, and can be converted to that of the difference-frequency problem by replacing ϕ by its complex conjugate. Expanding the right-hand side of (6.9) we obtain

$$\frac{\partial \varphi_B}{\partial n} = V_1+V_2+V_3, \tag{6.11a}$$

$$V_1 = b_1(\boldsymbol{x})\frac{\partial \phi_x}{\partial n}, \tag{6.11b}$$

$$V_2 = b_2(\boldsymbol{x})\frac{\partial \phi_y}{\partial n}, \tag{6.11c}$$

$$V_3 = b_3(\boldsymbol{x})\frac{\partial \phi_z}{\partial n}, \tag{6.11d}$$

where $b_j(\boldsymbol{x})$ are defined in (6.10). The substitution of the normal velocity components (6.11) into (6.8) will permit the reduction of the double spatial derivatives of ϕ to single derivatives. A vector theorem will be used due to Tuck (Ogilvie & Tuck 1969) derived for the evaluation of the hydrodynamic coefficients in ship motion theory. Its proof is repeated below and adapted to the present problem. Let $f(\boldsymbol{x})$ be a scalar differentiable function defined on the body boundary and in the fluid domain and let ϕ be a velocity potential. The variant of Stokes' theorem

$$\iint_{S_b} (\boldsymbol{n}\times\boldsymbol{\nabla})\times(f\boldsymbol{\nabla}\phi)\,\mathrm{d}s = \int_{C_w} \mathrm{d}\boldsymbol{l}\times\boldsymbol{\nabla}\phi\, f, \tag{6.12}$$

where C_w is the body waterline, is combined with the identity

$$(\boldsymbol{n}\times\boldsymbol{\nabla})\times(f\boldsymbol{\nabla}\phi) = f\boldsymbol{n}\times(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi)+f(\boldsymbol{n}\cdot\boldsymbol{\nabla})\,\boldsymbol{\nabla}\phi-\boldsymbol{n}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}f)+(\boldsymbol{n}\cdot\boldsymbol{\nabla}\phi)\,\boldsymbol{\nabla}f-f\boldsymbol{n}(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi). \tag{6.13}$$

The first and last terms in the right-hand side of (6.13) vanish identically by virtue of the irrotationality and incompressibility of the potential flow represented by ϕ. For wall-sided bodies which intersect the free surface at right angles, the vector $\mathrm{d}\boldsymbol{l}$ is horizontal and the following identity holds

$$\mathrm{d}\boldsymbol{l}\times\boldsymbol{\nabla}\phi = \boldsymbol{n}\phi_z-\boldsymbol{k}\frac{\partial\phi}{\partial n}. \tag{6.14}$$

Combining (6.13) and (6.14) with (6.12), we obtain the desired vector identity

$$\iint_{S_b} f(\boldsymbol{n}\cdot\boldsymbol{\nabla})\,\boldsymbol{\nabla}\phi = \iint_{S_b}\left[\boldsymbol{n}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}f)-\frac{\partial\phi}{\partial n}\boldsymbol{\nabla}f\right]\mathrm{d}s+\int_{C_w}\mathrm{d}l\,f\left(\boldsymbol{n}\phi_z-\boldsymbol{k}\frac{\partial\phi}{\partial n}\right). \tag{6.15}$$

Identity (6.15) will be used with $f = b_j \psi_i, j = 1, 2, 3$. Its x-component leads to

$$\iint_{S_b} b_1 \psi_i \frac{\partial \phi_x}{\partial n} = \iint_{S_b} [n_1(\nabla\phi \cdot \nabla f) - (\boldsymbol{n} \cdot \nabla\phi) f_x]\, \mathrm{d}s + \int_{C_w} \mathrm{d}l f n_1 \phi_z, \tag{6.16}$$

which allows the double spatial derivatives on the linear velocity potential ϕ to be replaced by single derivatives on ϕ and the auxiliary potential ψ_i. Combining (6.15) with (6.8)–(6.9) we obtain the similar replacement of all second spatial derivatives present in the second-order boundary condition (6.9).

The reciprocity relations derived in the present section do not entail any savings in computational effort but are more convenient to use since ψ_i can be determined as part of linear radiation problem. They resemble the reciprocity relations derived by Faltinsen & Loken (1978), Molin (1979) and Lighthill (1979) for the direct evaluation of the second-order forces which circumvent the determination of the second-order potential but require the evaluation of free-surface integrals involving products of the auxiliary potential and the forcing terms in the second-order free-surface condition.

7. Discussion and conclusions

A theory has been developed for the solution of the second-order surface wave radiation and diffraction problems around surface piercing bodies of arbitrary shape. Its principal element is the derivation of closed-form second-order Green functions which are used to construct an explicit particular solution satisfying the non-homogeneous free-surface condition. The body boundary condition is enforced by the addition of a homogeneous component subject to the linear free-surface condition and is determined from the solution of a linear problem.

The principal effort required for the implementation of the present theory is the evaluation of the second-order Green functions involved in the definitions of the particular solution by equations (5.4), (5.10) in the diffraction and (5.13), (5.14) in the radiation problems. The determination of the homogeneous component can be carried out easily using existing numerical techniques for the solution of the linear problem. The implementation of the present approach with existing panel methods requires only the discretization of the body boundary, unlike other solutions which discretize the free surface. Furthermore, the accuracy of the linear solution need not be any greater than is necessary in the linear problem, as long as the second-order Green functions are evaluated with adequate accuracy. By comparison, the evaluation of the slowly convergent free-surface integrals in existing numerical solutions hinges upon a highly accurate linear solution.

Given its robustness, the efficiency of the present method depends on the fast and accurate evaluation of the second-order Green functions. A similar statement holds true for the linear Green function when used for the solution of the linear problem by any of the boundary integral formulations presented in §4. Its inefficient evaluation is the principal reason why existing radiation-diffraction panel codes are often exercised in practice with unrealistically coarse discretizations. Its fast computation was the subject of a recent study by Newman (1985) which led to the development of efficient algorithms in water of infinite and finite depth. As in the linear case, the evaluation of the second-order Green functions is a task divorced from the shape of the body boundary or the number of panels required for the solution of the linear or second-order problems. It simply involves the numerical evaluation of explicit Fourier integrals and will be the subject of a future study.

The present theory applies both in two and three dimensions, and can be extended to water of finite depth by deriving the corresponding Green functions. In this case care must be exercised in the definition and interpretation of the second-order incident wave when the difference-frequency wavelength $2\pi g/(\omega_m - \omega_n)^2$ is large compared to the water depth. This problem has been addressed by Ogilvie (1983) and Agnon & Mei (1983). The third- and higher-order problems can also be treated along similar lines by deriving the corresponding Green functions.

The technique developed in this paper for the second-order surface-wave radiation and diffraction problems is applicable to a wider range of fluid mechanical problems with weak nonlinearities which can be treated by the perturbation method. More evident are extensions to nonlinear surface-wave problems which involve bodies undergoing a steady-state or unsteady translation near a free surface. Of related character is the problem of the nonlinear interaction of ambient surface waves with underwater sound radiated or scattered by bodies near a free surface.

In the problems discussed so far, the nonlinearity is present in a boundary condition rather than in the domain equation. Perhaps the most challenging flows in fluid mechanics are subject to nonlinear domain equations. For some of these problems, and under certain restrictions on the flow parameters, perturbation theory has proven useful (cf. Van Dyke 1975). An example is the slightly compressible isentropic flow past two- or three-dimensional bodies. The exact flow equation contains cubic nonlinear terms, functions of the velocity potential and its spatial derivatives. For sufficiently small Mach numbers in the steady-state case, these terms may be treated by the perturbation method starting either with the Laplace or with the Prandtl–Glauert equations in the linearized problem. The derivation of the cubic and higher-order Green functions would permit the solution of the corresponding problems around bodies of general shape by avoiding the discretization of the fluid domain.

Finally, non-homogeneous linear domain equations forced by lower-order solutions also arise in connection with the use of perturbation/matched-asymptotic expansions to flows past elongated lifting and non-lifting bodies. The application of the present method to such problems is at present less evident, but appears to deserve some study.

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